

On the set of equilibria of a bimatrix game

Citation for published version (APA):

Jansen, M. J. M., Jurg, P., & Vermeulen, A. J. (2002). On the set of equilibria of a bimatrix game: a survey. In S. Tijs, P. Born, & H. Peters (Eds.), *Chapters in Game Theory, Theory and Decision Library, series C* (Vol. 31, pp. 121-142). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47526-X_6

Document status and date:

Published: 01/01/2002

DOI:

[10.1007/0-306-47526-X_6](https://doi.org/10.1007/0-306-47526-X_6)

Document Version:

Publisher's PDF, also known as Version of record

Please check the document version of this publication:

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Chapter 6

On the Set of Equilibria of a Bimatrix Game: a Survey

BY MATHIJS JANSEN, PETER JURG, AND DRIES VERMEULEN

6.1 Introduction

Any survey on this topic should start with the celebrated results obtained by Nash. First of all he showed that every non-cooperative game in normal form has an equilibrium in mixed strategies (cf. Nash, 1950). He also established the well-known characterization of the equilibrium condition stating that a strategy profile is an equilibrium if and only if each player only puts positive weight on those pure strategies that are pure best responses to the strategies currently played by the other players (cf. Nash, 1951).

In the special case of matrix games the existence of equilibria was already established by von Neumann and Morgenstern (1944). Their results though show more than just that. They show for example that the collection of equilibria is a polytope. Furthermore they explain how one can use linear programming techniques to actually compute such an equilibrium.

Once the existence of equilibria was also established for bimatrix games, several authors, e.g. Vorobev, Kuhn, Mangasarian, Mills and Winkels, tried to develop methods based on linear programming to compute equilibria for bimatrix games. Later on authors like Winkels and Jansen also generalized the structure result and showed that the set of

equilibria of a bimatrix game can be written as the union of a finite number of polytopes. Such a representation of the set of equilibria is called a *decomposition* in this survey.

SEVEN PROOFS During the last few decades several different decompositions have been given. We will discuss seven of them and briefly comment on the differences and similarities between these decompositions. The first three can be seen as variations on the same line of reasoning. In this approach, first the extreme points of the polytopes involved in the decomposition of the equilibrium set are characterized. Subsequently an analysis is given of exactly how groups of extreme points generate one such polytope of the decomposition. We will first discuss these three methods.

(i) In the approach by Vorobev (1958) and Kuhn (1961) (as it is described in this survey) first a description is given of the collection \mathbb{IE}_1 of strategies of player 1 that can be combined to an extreme equilibrium. Then it is shown that

- (1) this collection is finite
- (2) the Cartesian product of the convex hull of a subset of \mathbb{IE}_1 with all strategies of player 2 that combine to an equilibrium with any one strategy of the subset in question is a polytope, and
- (3) any equilibrium is an element of such a product set.

(ii) Winkels (1979) basically uses the same steps in his proofs. The improvement over the proof of Vorobev and Kuhn is that the definition of the set \mathbb{IE}_1 is a bit different. This difference has the advantage that the proofs become shorter and more transparent.

(iii) Mangasarian's (1964) proof is based on a more symmetric treatment of the players. He looks at Cartesian products of subsets of \mathbb{IE}_1 with subsets of \mathbb{IE}_2 and shows that, whenever such a product is included in the equilibrium set, so will the convex hull of this product. Moreover, any one equilibrium is an element of the convex hull of at least one such a product.

The latter four proofs take what can be called a dual approach. Based on the characterization of the notion of an equilibrium in terms of carriers and best responses the defining systems of linear inequalities are given

directly. Subsequently it is shown that any solution of such a system is an equilibrium and that any equilibrium is indeed a solution of at least one of the systems generated by the approach in question.

(iv) The proof in Jansen (1981) is based on the two observations that any convex subset of the equilibrium set is contained in a maximal convex subset of the equilibrium set and that any maximal convex subset is a polytope. Thus, since each equilibrium by itself constitutes a convex subset of the equilibrium set, we again get the result that the equilibrium set is the union of polytopes. The fact that these polytopes are finite in number follows from the characterization of maximal convex subsets of the equilibrium set in terms of the carriers and best responses of the equilibria in such a subset.

(v) Undoubtedly the shortest proof is by Quintas (1989). He shows how to associate with each subset of the collection of pure strategies of player 1 and each subset of the collection of pure strategies of player 2 a polytope of equilibria. Since each equilibrium is evidently contained in such a polytope we easily get the result of Vorobev.

(vi) The approach of Jurg and Jansen (cf. Jurg, 1993) looks very much like the proof by Quintas. However, their approach yields a straightforward correspondence between the subsets of pure strategies used to generate the polytopes of the decomposition and faces of the equilibrium set.

(vii) The approach of Vermeulen and Jansen (1994) can be seen as a geometrical variation on the same theme. Its advantage though is that it can easily be adjusted to a proof of the same result for the collection of perfect and proper equilibria.

NEW ASPECTS Although this chapter is intended to be a survey, we would like to point out that we also used modern insights to get shorter or more transparent proofs of the original results. Further we used an idea of Winkels in order to show how the Mangasarian approach can be used to obtain the decomposition result. Finally we prove that the two decompositions of Vorobev and Winkels are in fact identical by showing that their (different) definitions of extreme strategies coincide.

Notation $\Delta_t = \{p \in \mathbb{R}^t \mid p \geq 0, \sum_{i=1}^t p_i = 1\}$. The unit vectors in \mathbb{R}^t are denoted by e_1, e_2, \dots, e_t . For $x, y \in \mathbb{R}^t$, we write $x \cdot y = \sum_{i=1}^t x_i y_i$. For $S \subset \mathbb{R}^t$ we denote by $\text{conv}(S)$ the convex hull of S and by $\text{cl}(S)$ the closure of S . For a convex set $C \subset \mathbb{R}^t$ we denote by $\text{relin}(C)$ the

relative interior of C and by $\text{ext}(C)$ the set of extreme points of C . For a finite set T , the collection of non-empty subsets of T is denoted by 2^T .

6.2 Bimatrix Games and Equilibria

An $m \times n$ -bimatrix game is played by two players, player 1 and player 2. Player 1 has a finite set $M = \{1, \dots, m\}$ and player 2 has a finite set $N = \{1, \dots, n\}$ of pure strategies. The payoff matrices $[a_{ij}]_{i \in M, j \in N}$ of player 1 and $[b_{ij}]_{i \in M, j \in N}$ of player 2 are denoted by A and B respectively. This game is denoted by (A, B) .

Now the game (A, B) is played as follows. Players 1 and 2 choose, independent of each other, a strategy $p \in \Delta_m$ and $q \in \Delta_n$, respectively. Here p_i (q_j) can be seen as the probability that player 1 (2) chooses his i -th row (j -th column). The (expected) payoff for player 1 is pAq and the expected payoff to player 2 is pBq .

A strategy pair $(p^*, q^*) \in \Delta_m \times \Delta_n$ is an *equilibrium* for the game (A, B) if

$$p^*Aq^* \geq pAq^* \text{ for all } p \in \Delta_m$$

and

$$p^*Bq^* \geq p^*Bq \text{ for all } q \in \Delta_n.$$

The set of all equilibria for the game (A, B) is denoted by $E(A, B)$. By a theorem of Nash (1950) this set is non-empty for all bimatrix games.

6.3 Some Observations by Nash

In a survey about equilibria it is inevitable to start with a description of concepts and results that can be found in John Nash' seminal paper 'Non-cooperative games' of 1951. Even in this first paper on the existence of equilibria Nash evidently realized that the key to a polyhedral description of the equilibrium set lies in the characterization of equilibria in terms of what are nowadays called carriers and best responses.

Since it is indeed the key to *all* known polyhedral descriptions of the Nash equilibrium set we will first have a look at his characterization of equilibria. It can be found at the bottom of page 287 of Nash's paper, but we will use the more modern terminology of Heuer and Millham (1976). Following them, we introduce for a strategy $p \in \Delta_m$ the *carrier* $C(p) = \{i \mid p_i > 0\}$ and the set $PB_2(p) = \{j \mid pBe_j \geq pBe_k \text{ for all } k\}$ of

pure best replies of player 2 to p . For a strategy $q \in \Delta_n$ the sets $C(q)$ and $PB_1(q)$ are defined in the same way.

Lemma 6.1 *Let (A, B) be a bimatrix game and let (p, q) be a strategy pair. Then $(p, q) \in E(A, B)$ if and only if $C(p) \subset PB_1(q)$ and $C(q) \subset PB_2(p)$.*

Proof. (a) If $(p, q) \in E(A, B)$, then

$$pAq = \sum_{i \in C(p)} p_i e_i Aq \leq \sum_{i \in C(p)} p_i \max_k e_k Aq = \max_k e_k Aq \leq pAq.$$

So $p_i > 0$ implies that $e_i Aq = \max_k e_k Aq$, that is: $C(p) \subset PB_1(q)$. Similarly, one shows that $C(q) \subset PB_2(p)$.

(b) If $C(p) \subset PB_1(q)$ and $C(q) \subset PB_2(p)$, then for all i

$$pAq = \sum_{\ell \in C(p)} p_\ell e_\ell Aq = \sum_{\ell \in C(p)} p_\ell \max_k e_k Aq = \max_k e_k Aq \geq e_i Aq.$$

Similarly, $pBq \geq pBe_j$ for all j . Hence, $(p, q) \in E(A, B)$. \square

Next we consider the concepts of interchangeability and sub-solutions introduced by Nash.

A subset S of the set of equilibria of a bimatrix game satisfies the *interchangeability* condition if for any pair $(p, q), (p', q') \in S$ we also have that $(p, q'), (p', q) \in S$.

If a subset S of the set of equilibria has the interchangeability property, then $S = S' \times S''$, where

$$S' = \{p \mid (p, q) \in S \text{ for some } q\}$$

and

$$S'' = \{q \mid (p, q) \in S \text{ for some } p\}$$

are called the *factor sets* of S . Since, obviously, a set $S = P \times Q$ of equilibria has the interchangeability property, sets of this form are precisely the sets with the interchangeability property. In Heuer and Millham (1976) sets of equilibria of the form $P \times Q$ were called *Nash sets*.

Nash used the term *sub-solution* for a Nash set that is not properly contained in another Nash set. In this chapter we prefer the term *maximal Nash set*, a term that was introduced by Heuer and Millham as well.

Nash gave a proof of the following result. Because it will be generalized in 6.15, the proof is left to the reader for now.

Lemma 6.2 *For a bimatrix game, a maximal Nash set is the product of two convex, compact sets.*

Finally, Nash proved that for a bimatrix game with only one maximal Nash set—he called such a game *solvable*—the set of equilibria is the product of two polytopes.

6.4 The Approach of Vorobev and Kuhn

In this section we will describe a result of Vorobev (1958) and its improved version of Kuhn (1961). Their method can be seen as a “one-sided” approach to the decomposition of the Nash equilibrium set into a finite number of (bounded) polyhedral sets.

They place themselves in the position of player 1. First they analyze which strategies of player 1 occur as extreme elements of certain polytopes of strategies that, combined with a finite number of strategies of player 2, are equilibria. This set of extreme elements is indicated by \mathbb{IE}_1 . Then they show that, for any subset P of \mathbb{IE}_1 , the collection $L(P)$ of strategies of player 2 that combine to an equilibrium for any element in P , is polyhedral. Finally they show that $\text{conv}(P) \times L(P)$, a polytope, is a subset of the Nash equilibrium set. Hence, since any equilibrium is indeed also an element of such a polytope, we get that the Nash equilibrium set is a, necessarily finite, union of polytopes.

In order to verify these claims, let (A, B) be an $m \times n$ -bimatrix game. For $p \in \Delta_m$ and $q \in \Delta_n$ we introduce the sets

$$L(p) = \{q' \in \Delta_n \mid (p, q') \in E(A, B)\}$$

and

$$K(q) = \{p' \in \Delta_m \mid (p', q) \in E(A, B)\}.$$

Since

$$K(q) = \Delta_m \cap \bigcap_{i=1}^m \{p \in \mathbb{R}^m \mid pAq \geq e_i Aq\} \cap \bigcap_{j=1}^n \{p \in \mathbb{R}^m \mid pBq \geq pBe_j\}$$

is the intersection of the bounded polyhedral set Δ_m and a finite number of halfspaces, $K(q)$ is a bounded polyhedral set. So $K(q)$ is a polytope. Similarly, $L(p)$ is a polytope.

For a set P of strategies of player 1 and a set Q of strategies of player 2, Vorobev introduces the sets

$$L(P) = \bigcap_{p \in P} L(p) \quad \text{and} \quad K(Q) = \bigcap_{q \in Q} K(q).$$

Obviously these sets are convex and compact.

Vorobev calls a strategy p of player 1 *extreme* if $p \in \text{ext}(K(Q))$ for some finite set Q of strategies of player 2. Let \mathbb{IE}_1 denote the set of extreme strategies of player 1.

In order to prove that \mathbb{IE}_1 is a finite set, Kuhn introduces the sets

$$P_B = \{(p, \beta) \mid pBe_j \leq \beta \text{ for all } j\}$$

and

$$Q_A = \{(q, \alpha) \mid e_i Aq \leq \alpha \text{ for all } i\}.$$

In words one could say that the set P_B is the collection of pairs (p, β) for which p is a strategy of player 1 and β is an upper bound on the payoffs player 2 can obtain given that player 1 plays p .

Since the sets P_B and Q_A are obviously polyhedral we can easily see that they only have a finite number of extreme points. Thus, the following lemma implies the finiteness of \mathbb{IE}_1 .

Lemma 6.3 *If $p \in \text{ext}(K(Q))$ for some finite set Q of strategies of player 2, then $(p, pBq) \in \text{ext}(P_B)$ for all $q \in Q$.*

Proof. Let $q \in Q$. Suppose that $(p, pBq) = \frac{1}{2}(p', \beta') + \frac{1}{2}(p'', \beta'')$, where $(p', \beta'), (p'', \beta'') \in P_B$. We have to prove that $(p', \beta') = (p'', \beta'')$. First we will show that $p', p'' \in K(Q)$. So let $q' \in Q$.

Since, for all j ,

$$pBe_j = \frac{1}{2}p'Be_j + \frac{1}{2}p''Be_j \leq \frac{1}{2}\beta' + \frac{1}{2}\beta'' = pBq,$$

we have for $j \in PB_2(p)$ that $p'Be_j = \beta'$. This implies that $\beta' = \max_k p'Be_k$. So $j \in PB_2(p')$. Furthermore, since $(p, q') \in E(A, B)$,

$$C(q') \subset PB_2(p) \subset PB_2(p'). \quad (1)$$

Since $p = \frac{1}{2}p' + \frac{1}{2}p''$,

$$C(p') \subset C(p) \subset PB_1(q'). \quad (2)$$

By (1) and (2), $(p', q') \in E(A, B)$. Similarly, $(p'', q') \in E(A, B)$. Hence, $p', p'' \in K(q')$.

Because $p \in \text{ext}(K(Q))$, this leads to the equality $p' = p''$. Since, for a $j \in PB_2(p)$, $\beta' = p'Be_j = p''Be_j = \beta''$, this proves that $(p, pBq) \in \text{ext}(P_B)$. \square

In a similar way one shows that, for a finite set P of strategies of player 1, the set $L(P)$ has a finite number of extreme points. Therefore the following theorem implies that the set of equilibria of a bimatrix game is the union of a finite number of polytopes.

Theorem 6.4 *For any bimatrix game (A, B)*

$$E(A, B) = \bigcup_{P \subset \mathbb{IE}_1} \text{conv}(P) \times L(P).$$

Proof. (a) Let P be a non-empty subset of \mathbb{IE}_1 such that $L(P) \neq \emptyset$. Suppose that $p \in \text{conv}(P)$ and $q \in L(P)$. Then $P \subset K(q)$ and the convexity of $K(q)$ implies that $p \in \text{conv}(P) \subset K(q)$. So $(p, q) \in E(A, B)$.

(b) Suppose that (p^*, q^*) is an element of $E(A, B)$. Then, by definition, the set $P = \text{ext}(K(q^*))$ is a subset of \mathbb{IE}_1 . Since $p^* \in K(q^*)$ and $K(q^*)$ is a polytope, $p^* \in K(q^*) = \text{conv}(\text{ext}(K(q^*))) = \text{conv}(P)$. Clearly, $(p, q^*) \in E(A, B)$ for all $p \in P$, that is: $q^* \in L(P)$. So $(p^*, q^*) \in \text{conv}(P) \times L(P)$. \square

Since, as we already observed, \mathbb{IE}_1 is a finite set, the above theorem immediately implies that the equilibrium set is the finite union of maximal Nash sets.

Another observation we would like to make at this point is that the previous approach also yields a way to index maximal Nash sets. This works as follows.

Lemma 6.5 *Let P be a set of strategies of player 1 and Q be a set of strategies of player 2. Then $P \times Q$ is a Nash set if and only if P is a subset of $K(Q)$ and Q is a subset of $L(P)$. It is a maximal Nash set if and only if P equals $K(Q)$ and Q equals $L(P)$.*

6.5 The Approach of Mangasarian and Winkels

In his proof Mangasarian (1964) also employs the polyhedral sets P_B and Q_A as they were introduced by Kuhn (1961). However, compared with the previous approach, Mangasarian's method of proof is based on a more symmetric treatment of the players.

Mangasarian proved that each equilibrium of a bimatrix game can be constructed by means of the finite set of extreme points of the two polyhedral sets P_B and Q_A corresponding to the game. In this section we will describe Mangasarian's ideas. Furthermore we will incorporate the concept of a Nash pair due to Winkels (1979) to show that for any bimatrix game the set of equilibria is the finite union of polytopes. The exposition of the proof we present here is a slightly streamlined version of the original proof by Winkels.

Mangasarian's approach is based on the following result, also proved by Mills (1960): a pair (p, q) of strategies is an equilibrium of a bimatrix game (A, B) if and only if there exist scalars α and β such that

$$\begin{aligned} (q, \alpha) &\in Q_A \\ (p, \beta) &\in P_B \\ pAq + pBq &= \alpha + \beta. \end{aligned}$$

Mangasarian calls a quartet (p, pBq, q, pAq) *extreme* if $(p, pBq) \in \text{ext}(P_B)$ and $(q, pAq) \in \text{ext}(Q_A)$. Obviously, in this case, (p, q) is an equilibrium. In order to prove that all equilibria can be found with the help of the finite number of extreme quartets, we need the following lemma due to Winkels (1979).

Lemma 6.6 *Let (p^*, q^*) be an equilibrium of a bimatrix game (A, B) , and let (p^*, p^*Bq^*) be a strict convex combination of pairs $(p^1, \beta_1), \dots, (p^r, \beta_r)$ in P_B . Then, for all i , (p^i, q^*) is an equilibrium of the game (A, B) and $\beta_i = p^iBq^*$.*

Proof. Suppose that $(p^*, p^*Bq^*) = \sum_k \lambda_k (p^k, \beta_k)$ where $\sum_k \lambda_k = 1$ and $\lambda_k > 0$ for all k .

Consider a strategy p^i . Then

$$C(p^i) \subset C(p^*) \subset PB_1(q^*) \quad (1)$$

and if $q_j^* > 0$, then

$$p^*Bq^* = p^*Be_j = \sum_k \lambda_k p^k Be_j \leq \sum_k \lambda_k \beta_k = p^*Bq^*.$$

Hence, $p^i B e_j = \beta_i$. This implies that $\beta_i = \max_k p^i B e_k$ so that

$$C(q^*) \subset P B_2(p^i). \quad (2)$$

In view of (1) and (2), $(p^i, q^*) \in E(A, B)$ and $\beta_i = p^i B q^*$. \square

The following result is due to Mangasarian (1964).

Theorem 6.7 *Let (p^*, q^*) be an equilibrium of a bimatrix game (A, B) . Then the quartet $(p^*, p^* B q^*, q^*, p^* A q^*)$ is a convex combination of extreme quartets.*

Proof. (a) First we will show that $(p^*, p^* B q^*)$ is a convex combination of extreme points of P_B .

Consider the linear function ℓ on P_B defined by

$$\ell(p, \beta) = p(A + B)q^* - p^* A q^* - \beta.$$

Then $\ell(p, \beta) \leq 0$ for any pair $(p, \beta) \in P_B$ and $(p^*, p^* B q^*)$ is an element of the compact, convex set

$$M = \{(p, \beta) \in P_B \mid \ell(p, \beta) = 0\}.$$

So, the theorem of Krein-Milman states that $(p^*, p^* B q^*)$ is a convex combination of elements of the set $\text{ext}(M)$. Since ℓ is a linear function, $\text{ext}(M) \subset \text{ext}(P_B)$. So $(p^*, p^* B q^*)$ is a convex combination of extreme points of P_B .

(b) According to part (a), we can write $(p^*, p^* B q^*)$ as a strict convex combination of pairs $(p^1, \beta_1), \dots, (p^r, \beta_r)$ in $\text{ext}(P_B)$. By Lemma 6.6, for all i , $(p^i, q^*) \in E(A, B)$ and $\beta_i = p^i B q^*$.

Similarly, we can write $(q^*, p^* A q^*)$ as a strict convex combination of pairs $(q^1, \alpha_1), \dots, (q^s, \alpha_s)$ in $\text{ext}(Q_A)$ such that, for all j , $(p^*, q^j) \in E(A, B)$ and $\alpha_j = p^* A q^j$.

The inclusion $C(q^j) \subset C(q^*)$ implies that $\beta_i = p^i B q^* = p^i B q^j$. Similarly, $\alpha_j = p^* A q^j = p^i A q^j$. So, for all i and j , $(p^i, p^i B q^j, q^j, p^i A q^j)$ is an extreme quartet. Since $(p^*, p^* B q^*, q^*, p^* A q^*)$ is a convex combination of the quartets $(p^i, p^i B q^j, q^j, p^i A q^j)$, the proof is complete. \square

Following Winkels we call a strategy p of player 1 *extreme* if there exists a strategy q of player 2 such that $(p, p B q, q, p A q)$ is an extreme quartet. Extreme strategies for player 2 are defined in a similar way. Let \mathbb{E}_i^* denote the (finite) set of extreme strategies of player i . Note that we

will show in Lemma 6.17 that the extreme strategies in the sense of Winkels coincide with the extreme strategies as introduced by Vorobev.

We call a pair (P, Q) with $P \subset \mathbb{IE}_1^*$ and $Q \subset \mathbb{IE}_2^*$ a *Nash pair* for the game (A, B) if $P \times Q$ is a Nash set.

Lemma 6.8 *If $P \times Q$ is a Nash set for a bimatrix game (A, B) , then $\text{conv}(P) \times \text{conv}(Q)$ is a Nash set too.*

Proof. If $(p, q) \in \text{conv}(P) \times \text{conv}(Q)$, then p and q are a convex combination of strategies $p_1, \dots, p_s \in P$ and $q_1, \dots, q_t \in Q$, respectively. Since $(p_i, q_j) \in P \times Q \subset E(A, B)$ for all j , $q_j \in L(p_i)$ for all j . By the convexity of $L(p_i)$, $q \in L(p_i)$. Hence $p_i \in K(q)$ for all i , which leads to $p \in K(q)$, that is: $(p, q) \in E(A, B)$. \square

Since $\mathbb{IE}_1^* \times \mathbb{IE}_2^*$ is a finite set, the number of Nash pairs is finite too. Furthermore, for every Nash pair (P, Q) , $\text{conv}(P) \times \text{conv}(Q) \subset E(A, B)$ and, by Theorem 6.7, each equilibrium is contained in a set $\text{conv}(P) \times \text{conv}(Q)$, where (P, Q) is a Nash pair. This proves that the set of equilibria of a bimatrix game is the finite union of polytopes.

Theorem 6.9 *For any bimatrix game (A, B)*

$$E(A, B) = \bigcup_{(P, Q) \text{ is a Nash pair}} \text{conv}(P) \times \text{conv}(Q).$$

Note that, due to the definition of a Nash pair, not all Nash sets used in this decomposition are necessarily maximal. Thus, some of them may be redundant.

6.6 The Approach of Winkels

In this section we will describe the result of Vorobev and Kuhn again. This time we will follow the ideas developed by Winkels (1979) by using his definition of an extreme strategy of a player. Winkels came to his definition by combining the ideas of Mangasarian and Kuhn.

Lemma 6.10 *If P is a set of strategies of player 1, then*

- (a) $P \subset K(L(P))$ if $L(P) \neq \emptyset$
- (b) $L(\text{conv}(P)) = L(P)$.

Proof. We will give a proof of part (b) only. Because $P \subset \text{conv}(P)$, $L(P) \supset L(\text{conv}(P))$.

Now suppose that $q \in L(P)$ and that p is a convex combination of strategies $p^i \in P$. Then the convexity of $K(q)$ implies that $p \in K(q)$. That is: (p, q) is an equilibrium. This proves that $q \in L(\text{conv}(P))$. \square

Theorem 6.11 stated below is Winkels' version of Vorobev's result. In fact, by Lemma 11, this theorem is identical to (Vorobev's) Theorem 6.4.

Theorem 6.11 *For any bimatrix game (A, B)*

$$E(A, B) = \bigcup_{P \subset \mathbb{E}_1^*} \text{conv}(P) \times L(P).$$

Proof. (a) Let P be a non-empty subset of \mathbb{E}_1^* such that $L(P) \neq \emptyset$. According to Lemma 6.10(a), $P \subset K(L(P))$, whereas the convexity of the right-hand set implies that $\text{conv}(P) \subset K(L(P))$. In combination with Lemma 6.10(b) and Lemma 6.5 this inclusion proves that

$$\text{conv}(P) \times L(P) \subset E(A, B).$$

(b) In order to prove the converse inclusion, assume that $(p, q) \in E(A, B)$. According to Theorem 6.7, the quartet (p, pBq, q, pAq) is a strict convex combination of extreme quartets, say $(p^1, \beta_1, q^1, \alpha_1), \dots, (p^r, \beta_r, q^r, \alpha_r)$. Now let $P = \{p^1, \dots, p^r\}$. Then $P \subset \mathbb{E}_1^*$ and $p \in \text{conv}(P)$. By Lemma 6.6, $q \in L(p^i)$ for all i which implies that $q \in L(P)$. Hence, (p, q) is an element of $\text{conv}(P) \times L(P)$, and the proof is complete. \square

In order to prove that the sets described in the foregoing theorem are in fact polytopes, Winkels introduces for a subset P of \mathbb{E}_1^* the finite set

$$L^{\text{ext}}(P) = \{q \in \mathbb{E}_2^* \mid (p, q) \in E(A, B) \text{ for all } p \in P\}$$

and he concludes that $L(P)$ is a polytope on the basis of the following result.

Lemma 6.12 *If P is a subset of \mathbb{E}_1^* , then $L(P) = \text{conv}(L^{\text{ext}}(P))$.*

Proof. Since $L^{\text{ext}}(P) \subset L(P)$ and $L(P)$ is convex, $\text{conv}(L^{\text{ext}}(P)) \subset L(P)$.

Now let $q \in L(P)$ and $\tilde{p} \in P$. As in the proof of Theorem 6.11 one shows that a set $Q \subset \mathbb{E}_2^*$ exists such that $(\tilde{p}, q) \in K(Q) \times \text{conv}(Q)$.

Since $(p, q) \in E(A, B)$ for all $p \in P$, Lemma 6.6 implies that, for any $q^j \in Q$, $(p, q^j) \in E(A, B)$ for all $p \in P$. Therefore $q^j \in L^{\text{ext}}(P)$ for all $q^j \in Q$. Since $q \in \text{conv}(Q)$, $q \in \text{conv}(L^{\text{ext}}(P))$. \square

6.7 The Approach of Jansen

In the approaches described in the foregoing two sections, extreme strategies were the central issue. In the work of Jansen (1981) though the starting point was the notion of a maximal Nash set. In fact the source of inspiration for the research of Jansen was Heuer and Millham (1976), where several properties of (the intersection of) these maximal Nash sets were obtained.

Lemma 6.8 states in fact that any Nash set is contained in a convex Nash set. As a consequence of this result, a maximal Nash set is a convex set. Before we can show that the maximal Nash sets are in fact the maximal convex sets, we first need a lemma.

Lemma 6.13 *Any convex subset C of the set of equilibria of a bimatrix game (A, B) is contained in a (convex) Nash set.*

Proof. Assume that $(p, q), (p', q') \in C$. We will show that (p, q') and (p', q) are equilibria.

Consider, for $t \in (0, 1)$, the strategies $p(t) = tp + (1 - t)p'$ and $q(t) = tq + (1 - t)q'$. Since $(p(t), q(t)) \in C \subset E(A, B)$, for t close to 1

$$C(p) \cup C(p') = C(p(t)) \subset PB_1(q(t)) \subset PB_1(q).$$

So, $C(p') \subset PB_1(q)$. Similarly, for t close to 0

$$C(q) \cup C(q') = C(q(t)) \subset PB_2(p(t)) \subset PB_2(p')$$

and therefore $C(q) \subset PB_2(p')$. Hence, $(p', q) \in E(A, B)$. Similarly, $(p, q') \in E(A, B)$. This proves that

$$\{p \mid \text{there is a } q \text{ with } (p, q) \in C\} \times \{q \mid \text{there is a } p \text{ with } (p, q) \in C\}$$

is a (convex) Nash set containing C . □

Theorem 6.14 *Let C be a convex subset of the set of equilibria of a bimatrix game (A, B) . Then C is a maximal convex subset if and only if C is a maximal Nash set.*

Proof. (a) Suppose that C is a maximal convex subset of $E(A, B)$. Then according to Lemma 6.13, C is contained in and hence equal to a convex Nash set. In view of Lemma 6.8, this Nash set must be maximal.

(b) Let C be a maximal Nash set and suppose that C is contained in the convex set T . According to Lemma 6.13, T is contained in a Nash set, say T' . So, by the maximality of C , this is possible only if $C = T = T'$. Hence, C is a maximal convex set. \square

If (p, q) is an equilibrium of a bimatrix game (A, B) , then $\{(p, q)\}$ is a convex subset of $E(A, B)$. Hence we can find, applying Zorn's Lemma, a maximal convex subset of $E(A, B)$ containing (p, q) . In view of Theorem 6.14, each equilibrium of the game (A, B) is contained in a maximal Nash set and $E(A, B)$ is the union of such sets. In order to show that the number of maximal Nash sets is finite, we need the following lemma.

Lemma 6.15 *Let $S = S_1 \times S_2$ be a maximal Nash set for a bimatrix game (A, B) . Further, let $(\tilde{p}, \tilde{q}) \in \text{relint}(S)$ and let (p, q) be a strategy pair. Then*

$$(a) \quad S = K(\tilde{q}) \times L(\tilde{p})$$

$$(b) \quad (p, q) \in S \text{ if and only if } C(p) \subset C(\tilde{p}), C(q) \subset C(\tilde{q}), PB_1(q) \supset PB_1(\tilde{q}) \text{ and } PB_2(p) \supset PB_2(\tilde{p}).$$

Proof. (a) Obviously, $S \subset S_1 \times L(\tilde{p})$. In order to show that $S_1 \times L(\tilde{p})$ is a Nash set, suppose that $(p, q) \in S_1 \times L(\tilde{p})$. Since $\tilde{p} \in \text{relint}(S_1)$, there exists a $t > 1$ such that $p(t) = (1 - t)p + t\tilde{p} \in S_1$. Then $C(p) \subset C(\tilde{p})$. Since $(p(t), \tilde{q}), (p, \tilde{q}) \in S$, for $j \in PB_2(\tilde{p})$

$$\begin{aligned} \tilde{p}B\tilde{q} = \tilde{p}Be_j &= t^{-1}p(t)Be_j + (1 - t^{-1})pBe_j \\ &\leq t^{-1}p(t)B\tilde{q} + (1 - t^{-1})pB\tilde{q} = \tilde{p}B\tilde{q}. \end{aligned}$$

This proves that $pBe_j = pB\tilde{q}$ and, since $(p, \tilde{q}) \in E(A, B)$, that $j \in PB_2(p)$.

Thus we may conclude that $PB_2(\tilde{p}) \subset PB_2(p)$. This implies, in combination with the fact that $C(p) \subset C(\tilde{p})$ and $(\tilde{p}, q) \in E(A, B)$, that

$$C(p) \subset C(\tilde{p}) \subset PB_1(q) \quad \text{and} \quad C(q) \subset PB_2(\tilde{p}) \subset PB_2(p).$$

So $(p, q) \in E(A, B)$, which proves that $S_1 \times L(\tilde{p})$ is a Nash set containing S . Since S is maximal, $S_2 = L(\tilde{p})$. In a similar manner one shows that $S_1 = K(\tilde{q})$.

(b) In part (a) it has been proved that the four inclusions mentioned in the theorem hold for a $(p, q) \in S$. If, on the other hand, the four

inclusions hold, then it follows that $(p, \hat{q}), (\hat{p}, q) \in E(A, B)$. This implies that $p \in K(\hat{q})$ and $q \in L(\hat{p})$, that is: $(p, q) \in K(\hat{q}) \times L(\hat{p}) = S$. \square

By Lemma 6.15 a maximal Nash set is completely determined by the quartet $(C(\hat{p}), PB_2(\hat{p}), C(\hat{q}), PB_1(\hat{q}))$, where (\hat{p}, \hat{q}) is some equilibrium in its relative interior. Since there is only a finite number of such quartets, we obtain the following result of Jansen (1981).

Theorem 6.16 *The set of equilibria of a bimatrix game is a (not necessarily disjoint) union of a finite number of maximal Nash sets.*

Finally we will show that the extreme strategies as introduced by Winkels coincide with the extreme strategies in the sense of Vorobev.

Lemma 6.17 *For a strategy p of player 1 the following statements are equivalent:*

- (1) *there exist a strategy q of player 2 and a maximal Nash set S such that $(p, q) \in \text{ext}(S)$*
- (2) $p \in \mathbb{IE}_1$
- (3) $p \in \mathbb{IE}_1^*$.

Proof. We will prove the implications $(1) \implies (2) \implies (3) \implies (1)$.

(a) Suppose that $(p, q) \in \text{ext}(S)$ for some strategy q of player 2 and some maximal Nash set S . By Lemma 6.15, $p \in \text{ext}(K(\hat{q}))$ and $q \in \text{ext}(L(\hat{p}))$, where $(\hat{p}, \hat{q}) \in \text{relint}(S)$. Hence, $p \in \mathbb{IE}_1$.

(b) Suppose that $p \in \mathbb{IE}_1$. Let $q \in \mathbb{IE}_2$. Then finite sets P and Q of strategies of player 1 and 2 exist such that $p \in \text{ext}(K(Q))$ and $q \in \text{ext}(L(P))$. In view of Lemma 6.3, this implies that $(p, \beta) \in \text{ext}(P_B)$ for some β and $(q, \alpha) \in \text{ext}(Q_A)$ for some α . Since $(p, q) \in E(A, B)$, $\beta = pBq$ and $\alpha = pAq$. So (p, pBq, q, pAq) is an extreme quartet, that is: $(p, q) \in \mathbb{IE}_1^* \times \mathbb{IE}_2^*$. So $p \in \mathbb{IE}_1^*$.

(c) Suppose that $p \in \mathbb{IE}_1^*$. By definition (of \mathbb{IE}_1^*) there is a strategy q in \mathbb{IE}_2^* such that $(p, q) \in E(A, B)$. Then $(p, q) \in S$ for some maximal Nash set S . If $(p, q) \notin \text{ext}(S)$, then there exist $(p', q'), (p'', q'') \in S$ such that $(p, q) = \frac{1}{2}(p', q') + \frac{1}{2}(p'', q'')$ and $p' \neq p''$ or $q' \neq q''$. Let $p' \neq p''$. Then $(p', q), (p'', q) \in S$ so that $(p', p'Bq)$ and $(p'', p'Bq)$ are elements of P_B . Since $(p, pBq) = \frac{1}{2}(p', p'Bq) + \frac{1}{2}(p'', p'Bq)$, this contradicts the fact that $(p, pBq) \in \text{ext}(P_B)$. \square

A similar result holds for strategies of player 2.

6.8 The Approach of Quintas

A very short and straightforward proof is the following one by Quintas (1989). With each set I of pure strategies of player 1 and set J of pure strategies of player 2 he associates the collection of strategy pairs (p, q) such that the carrier of p is contained in I , all pure strategies in J are best responses to p , the carrier of q is contained in J and all pure strategies in I are best responses to q . It is straightforward that such a collection is a polytope, that there is only a finite number of them, and that each equilibrium is contained in such a polytope.

More formally, for an $m \times n$ bimatrix game (A, B) and a pair $(I, J) \in 2^M \times 2^N$, Quintas introduces the subset

$$H(I, J) = \{(p, q) \in \Delta_m \times \Delta_n \mid C(p) \subset I \subset PB_1(q) \\ \text{and } C(q) \subset J \subset PB_2(p)\}$$

of $E(A, B)$. Because this set is bounded and determined by finitely many inequalities, it is a polytope.

If, for an equilibrium (p, q) , we take $I = PB_1(q)$ and $J = PB_2(p)$, then obviously $(p, q) \in H(I, J)$. So

$$E(A, B) = \bigcup_{(I, J) \in 2^M \times 2^N} H(I, J).$$

One can show that for a pair $(I, J) \in 2^M \times 2^N$ the polytope $H(I, J)$ is a face of a maximal Nash set. However, generally there is not a nice relation between elements of $2^M \times 2^N$ and faces of maximal Nash sets as can be seen by considering the game

$$(A, B) = \begin{bmatrix} (2, 1) & (1, 0) & (1, 1) \\ (2, 0) & (1, 1) & (0, 0) \end{bmatrix}.$$

Although $((\frac{1}{2}, \frac{1}{2}), e_1)$ is an extreme equilibrium of this game (and hence a face of some maximal Nash set), there is no pair $(I, J) \in 2^{\{1,2\}} \times 2^{\{1,2,3\}}$ such that $H(I, J) = \{((\frac{1}{2}, \frac{1}{2}), e_1)\}$.

Moreover for this game $H(\{1, 2\}, \{1, 2\}) = H(\{1, 2\}, \{1, 2, 3\})$. In the next section we will describe an approach not suffering from this drawback.

6.9 The Approach of Jurg and Jansen

In this section we describe the approach of Jurg and Jansen (cf. Jurg, 1993) who adapted the method of Quintas by replacing the pairs he

dealt with by quartets consisting of the two carriers and the two sets of pure best replies of a strategy pair. Their approach reveals more of the structure of the set of equilibria and in particular of maximal Nash sets.

By Lemma 6.1 a strategy pair (p, q) is an equilibrium of an $m \times n$ -bimatrix game (A, B) if and only if the (equilibrium) inclusions $C(p) \subset PB_1(q)$ and $C(q) \subset PB_2(p)$ are satisfied. To check this relation we need the quartet

$$(C(p), PB_2(p), C(q), PB_1(q)).$$

If (p, q) is an equilibrium of (A, B) , then this quartet is called the *characteristic quartet* of (p, q) . The set of all characteristic quartets for the bimatrix game (A, B) is denoted by $Char(A, B)$. Clearly, as a subset of $2^M \times 2^N \times 2^N \times 2^M$, this set is finite and it partitions the set of equilibria.

For a quartet $(I, J, K, L) \in Char(A, B)$ the set $F(I, J, K, L)$ is the collection of pairs of strategies (p, q) for which

$$C(p) \subset I, PB_2(p) \supset J, C(q) \subset K \text{ and } PB_1(q) \supset L\},$$

and it is called the *characteristic set* corresponding to this quartet. If (p, q) is an element of $F(I, J, K, L)$, then

$$C(p) \subset I \subset L \subset PB_1(q)$$

and

$$C(q) \subset K \subset J \subset PB_2(p),$$

which implies that (p, q) satisfies the equilibrium inclusions. Hence, $F(I, J, K, L)$ is a subset of $E(A, B)$. Clearly an equilibrium (p, q) is contained in the characteristic set corresponding to the characteristic quartet of (p, q) , so we have

$$E(A, B) = \bigcup_{(I, J, K, L) \in Char(A, B)} F(I, J, K, L).$$

Since there are only finitely many different characteristic quartets, there are also finitely many different characteristic sets. Again, each characteristic set is bounded and described by finitely many linear inequalities and therefore a polytope. Hence

Theorem 6.18 *The equilibrium set of a bimatrix game is the union of a finite number of polytopes.*

Because of this finite number, we can assume that in Theorem 6.18 each of the polytopes or equivalently each of the characteristic sets is *maximal*, i.e. not properly contained in another one.

One easily checks that a characteristic set is a Nash set. Moreover

Theorem 6.19 *Let (A, B) be a bimatrix game. A maximal characteristic set is a maximal Nash set for (A, B) and vice versa.*

Proof. We have proved the theorem if we show that each Nash set is contained in a characteristic set.

Let T be a Nash set. According to Lemma 6.8, $S = \text{conv}(T)$ is also a Nash set.

Let $(\tilde{p}, \tilde{q}) \in \text{relint}(S)$. As in part (a) of the proof of Lemma 6.15, one can show that for a $(p, q) \in T$, $C(p) \subset C(\tilde{p})$, $PB_2(\tilde{p}) \subset PB_2(p)$, $C(q) \subset C(\tilde{q})$ and $PB_1(\tilde{q}) \subset PB_1(q)$. Hence (p, q) is an element of the characteristic set corresponding to the characteristic quartet of (\tilde{p}, \tilde{q}) . By consequence, T is contained in this characteristic set. \square

Thus Theorem 6.19 settles the existence of maximal Nash sets. Furthermore, this theorem implies Theorem 6.16. Note that in this approach Zorn's lemma is not used.

Obviously, a characteristic set $F(I, J, K, L)$ is maximal if and only if there is no characteristic quartet (I', J', K', L') different from (I, J, K, L) such that $I \subset I', J \supset J', K \subset K'$ and $L \supset L'$. Hence the following lemma implies that, more generally, each characteristic set is a face of a maximal Nash set and conversely.

Lemma 6.20 *Let (I, J, K, L) be a characteristic quartet for a game (A, B) . Then F is a face of $F(I, J, K, L)$ if and only if $F = F(I', J', K', L')$ for some characteristic quartet with $I' \subset I, J' \supset J, K' \subset K$ and $L' \supset L$.*

Proof. (a) First let (I', J', K', L') be a characteristic quartet such that $I' \subset I, J' \supset J, K' \subset K$ and $L' \supset L$. Then $F(I', J', K', L') \subset F(I, J, K, L)$. Let G be the smallest face of $F(I, J, K, L)$ containing $F(I', J', K', L')$. We will prove that $F(I', J', K', L')$ is a face of $F(I, J, K, L)$ by showing that $G \subset F(I', J', K', L')$.

Since $F(I', J', K', L') \cap \text{relint}(G) \neq \emptyset$, we can take a $(\tilde{p}, \tilde{q}) \in F(I', J', K', L') \cap \text{relint}(G)$. Let $(p, q) \in G$. Arguments similar to those in the proof of Theorem 6.19 yield that $C(p) \subset C(\tilde{p})$,

$PB_2(\tilde{p}) \subset PB_2(p)$, $C(q) \subset C(\tilde{q})$ and $PB_1(\tilde{q}) \subset PB_1(q)$. Since moreover $(\tilde{p}, \tilde{q}) \in F(I', J', K', L')$, it follows that $(p, q) \in F(I', J', K', L')$. So $G \subset F(I', J', K', L')$.

(b) Secondly let F be a face of $F(I, J, K, L)$. Choose $(\tilde{p}, \tilde{q}) \in \text{relint}(F)$. As in the foregoing part one can show that $F \subset F(I', J', K', L')$, where $I' = C(\tilde{p}) \subset I$, $J' = PB_2(\tilde{p}) \supset J$, $K' = C(\tilde{q}) \subset K$ and $L' = PB_1(\tilde{q}) \supset L$.

The proof is complete if we can show that $F = F(I', J', K', L')$. Therefore we suppose that $F \neq F(I', J', K', L')$.

By part (a), $F(I', J', K', L')$ is a face of $F(I, J, K, L)$. Hence, F is a face of $F(I', J', K', L')$.

Choose $(p', q') \in \text{relint}(F(I', J', K', L'))$. It is easily shown that (I', J', K', L') is the characteristic quartet of (p', q') .

Let, for $\epsilon > 0$,

$$(p^\epsilon, q^\epsilon) = \frac{1}{1-\epsilon}(\tilde{p}, \tilde{q}) - \frac{\epsilon}{1-\epsilon}(p', q').$$

Then $(p^\epsilon, q^\epsilon) \in F(I', J', K', L')$ for small ϵ .

Since F is a face of $F(I', J', K', L') \neq F$, there are a pair $(u, v) \in \mathbb{R}^m \times \mathbb{R}^n$ and a real number c such that

$$(p, q) \cdot (u, v) = c, \text{ for } (p, q) \in F$$

and

$$(p, q) \cdot (u, v) < c, \text{ for } (p, q) \in F(I', J', K', L') \setminus F.$$

This implies that

$$c \geq (p^\epsilon, q^\epsilon) \cdot (u, v) = \frac{1}{1-\epsilon}(\tilde{p}, \tilde{q}) \cdot (u, v) - \frac{\epsilon}{1-\epsilon}(p', q') \cdot (u, v) > c,$$

which is a contradiction. Hence $F = F(I', J', K', L')$. \square

In fact, since $F(I, J, K, L) = F(I', J', K', L')$ implies that (I, J, K, L) equals (I', J', K', L') , we infer from Lemma 6.20:

Theorem 6.21 *For a bimatrix game (A, B) there is a one-to-one correspondence between the elements of $\text{Char}(A, B)$ and the set of faces of maximal Nash sets for (A, B) .*

6.10 The Approach of Vermeulen and Jansen

In this section the method of Vermeulen and Jansen (1994) is described. The advantage of this method is that it can easily be adjusted to get the same structure result for perfect (cf. Vermeulen and Jansen, 1994) and proper equilibrium (cf. Jansen, 1993).

The key of this approach is the introduction of an equivalence relation for each player by identifying the strategies to which the other player has the same pure best replies. With the help of these relations the strategy spaces of both players are partitioned in a finite number of equivalence classes. The closure of each of these classes appears to be a polytope. By considering the intersection of the set of equilibria with the closure of the product of two equivalence classes (one for each player), Vermeulen and Jansen show that the set of equilibria is in fact the finite union of polytopes.

For a bimatrix game two strategies p and \tilde{p} are called *best-reply equivalent*, denoted as $p \sim_{BR} \tilde{p}$, if $PB_2(p) = PB_2(\tilde{p})$. In a similar way an equivalence relation can be defined for the strategies of player 2.

Since for an $m \times n$ -bimatrix game, $PB_2(p)$ is a subset of N for all $p \in \Delta_m$, the number of equivalence classes in Δ_m corresponding to the equivalence relation \sim_{BR} must be finite. The equivalence classes are denoted as $\mathcal{V}_1, \dots, \mathcal{V}_S$. Similarly, Δ_n is the finite union of equivalence classes, say $\mathcal{W}_1, \dots, \mathcal{W}_T$. For later purposes, we choose representants p^s in \mathcal{V}_s and q^t in \mathcal{W}_t for all s and t .

Obviously, each equivalence class is a convex set. Furthermore,

Lemma 6.22 *For all pairs (s, t)*

$$\text{cl}(\mathcal{V}_s) = \{p \in \Delta_m \mid PB_2(p) \supset PB_2(p^s)\}$$

and

$$\text{cl}(\mathcal{W}_t) = \{q \in \Delta_n \mid PB_1(q) \supset PB_1(q^t)\}.$$

Proof. We will only give a proof of the first equality.

Obviously, for a $p \in \text{cl}(\mathcal{V}_s)$, $PB_2(p) \supset PB_2(p^s)$.

For a $p \in \Delta_m$ with $PB_2(p) \supset PB_2(p^s)$, we consider the strategy

$$p(\epsilon) = \epsilon p + (1 - \epsilon)p^s,$$

where $0 \leq \epsilon < 1$. In order to show that $p(\epsilon) \in \mathcal{V}_s$ for all ϵ , first we take a $k \in PB_2(p^s) \subset PB_2(p)$. Then for all j

$$p(\epsilon)Be_k = \epsilon pBe_k + (1 - \epsilon)p^sBe_k \geq \epsilon pBe_j + (1 - \epsilon)p^sBe_j = p(\epsilon)Be_j.$$

For a $k \notin PB_2(p^s)$ and a $j \in PB_2(p^s) \subset PB_2(p)$

$$p(\epsilon)Be_k = \epsilon pBe_k + (1 - \epsilon)p^sBe_k < \epsilon pBe_j + (1 - \epsilon)p^sBe_j = p(\epsilon)Be_j.$$

Hence, for all ϵ , $PB_2(p(\epsilon)) = PB_2(p^s)$ which means that $p(\epsilon) \in \mathcal{V}_s$. Then however

$$p = \lim_{\epsilon \rightarrow 1} p(\epsilon) \in \text{cl}(\mathcal{V}_s)$$

which concludes the proof. \square

With the help of the representation of the closure of an equivalence class as given in the previous lemma, it is easy to prove that the closure of an equivalence class corresponding to the relation \sim_{BR} is a polytope.

Next we consider the set of equilibria contained in the closure of the product of two equivalence classes (one for each player). For a pair (s, t) , we consider the Nash set

$$\mathcal{N}_{s,t} = \{(p, q) \in \text{cl}(\mathcal{V}_s) \times \text{cl}(\mathcal{W}_t) \mid C(p) \subset PB_1(q^t) \text{ and } C(q) \subset PB_2(p^s)\}.$$

Obviously a Nash set $\mathcal{N}_{s,t}$ is a polytope and each equilibrium is contained in some Nash set $\mathcal{N}_{s,t}$. Further, if (p, q) is an element of some Nash set \mathcal{N}_{st} , then Lemma 6.22 implies that $PB_1(q) \supset PB_1(q^t) \supset C(p)$ and $PB_2(p) \supset PB_2(p^s) \supset C(q)$. Hence, by Lemma 6.1, (p, q) is an equilibrium. So we have the following result.

Theorem 6.23 *The set of equilibria of a bimatrix game is the finite union of polytopes.*

Since the number of Nash sets $\mathcal{N}_{s,t}$ is finite, each Nash set is contained in a maximal one and the set of equilibria of a bimatrix game is the finite union of maximal Nash sets.

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